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THE STABILITY OF THE SOLUTIONS OF SOME BOUNDARY-VALUE PROBLEMS FOR HYPERBOLIC EQUATIONS†

S. A. YEGORUSHKIN and A. G. KULIKOVSKII

Moscow

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The behaviour of the solutions of linear hyperbolic equations is investigated as $t\to\infty$ in the half-space $x>0$, $-\infty<y_\alpha<\infty$, $\alpha=1, 2, \ldots, r$, with boundary conditions defined on the boundary $x=0$. The equations and the boundary condition are assumed to be homogeneous with respect to the order of differentiation and all coefficients are assumed to be constant. A problem of this type has been previously studied in detail in connection with the stability of shock waves in gas dynamics [1–4] and some particular results have also been obtained for magnetohydrodynamic shocks [5–7].

In general, as will be shown below, the disturbances may have the same types of behaviour as $t\to\infty$ as in [2, 4]: the disturbances increase exponentially (instability), decay as a power function (stability), or remain bounded (neutral stability). The transitions of the system to an unstable, stable, and neutrally stable state are investigated and the criteria for these transitions are derived. These criteria are used to establish the existence of neutrally stable magnetohydrodynamic shocks even in the case of an ideal gas, a phenomenon that has not been previously documented [5–7]. The existence of an *a priori* bound on the solution has been proved for these systems in cases of stability and neutral stability [8, 9].

The interaction of disturbances with the boundary in the case of neutral stability produces a non-smooth solution, so that the *a priori* bound of [9] is unimprovable. An elementary explanation of this effect is proposed. It is shown that the addition of small non-differential terms to the equations and the boundary conditions does not cause the problems to become ill-posed if the parameters of the original problem ensure neutral stability.

The behaviour of disturbances on the boundary of the half-space is described by the solution of the Cauchy problem for some systems of partial differential equations of a high order with special conditions on the external forces and the initial values. This result is similar to that observed in gas dynamics [10].

The stability of solutions with boundary conditions at $x=0$ for $x>0$ and $x<0$ is analysed similarly and does not require a separate treatment.

1. FORMAL CONSTRUCTION OF THE FOURIER–LAPLACE SOLUTION

FOR A system of linear hyperbolic equations with constant coefficients

$$\mathbf{E}\frac{\partial \mathbf{U}}{\partial t}+\mathbf{A}\frac{\partial \mathbf{U}}{\partial x}+\mathbf{B}_\alpha\frac{\partial \mathbf{U}}{\partial y_\alpha}=0, \qquad \alpha=1, 2, \ldots r \tag{1.1}$$

($\mathbf{U}$ is a column vector and $\mathbf{E}$, $\mathbf{A}$, $\mathbf{B}$ are $n\times n$ matrices) in the half-space $x>0$, $-\infty<y_\alpha<\infty$, consider the mixed boundary-value problem

$$t=0: \mathbf{U}=\mathbf{U}_0(x, y_\alpha) \tag{1.2}$$

$$x=0, \quad t>0: \mathbf{E}_1\frac{\partial U}{\partial t}+\mathbf{C}\frac{\partial \mathbf{U}}{\partial x}+\mathbf{D}_\alpha\frac{\partial \mathbf{U}}{\partial y_\alpha}=0$$

$$t>0, \quad x\to\infty: |\mathbf{U}|<\infty$$

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Here $\mathbf{E}_1$, $\mathbf{C}$ and $\mathbf{D}_\alpha$ are constant $m \times n$ matrices, $m \leqslant n$. The number m of boundary conditions for $x = 0$ is chosen so that the problem is well-posed [11] (see below).

We will solve problem (1.1), (1.2) by applying a Fourier transformation to the function $\mathbf{U}(x, y_\alpha, t)$ with respect to the variables y_α and a Laplace transformation with respect to the variable t. Thus

$$\mathbf{U}^L(x, z, k_\alpha) = \frac{1}{(2\pi)^r} \int_{-\infty}^{\infty} e^{-ik_\alpha y_\alpha}\, dy_1 \ldots dy_r \frac{1}{2\pi i} \int_{-\infty}^{\infty} \mathbf{U}(x, y_\alpha, t)\, e^{-zt}\, dt$$

Setting $z = -ik\Omega$, $k = (k_1^2 + k_2^2 + \ldots + k_r^2)^{1/2}$ (Ω is the phase velocity of propagation of waves in the direction of the vector $\mathbf{k}$ perpendicular to the x-axis), we rewrite problem (1.1), (1.2) in the form

$$\mathbf{A}\frac{d\mathbf{U}^L}{dx} + ik(\mathbf{B} - \Omega\mathbf{E})\,\mathbf{U}^L = \mathbf{E}\mathbf{U}_0^F(x, k_\alpha), \quad \left(\mathbf{B} = \frac{k_\alpha}{k}\mathbf{B}_\alpha\right) \tag{1.3}$$

$$x = 0: \mathbf{C}\frac{d\mathbf{U}^L}{dx} + ik(\mathbf{D} - \Omega\mathbf{E}^1)\,\mathbf{U}^L = \mathbf{E}^1\mathbf{U}_0^F(0, k_\alpha), \quad \left(\mathbf{D}_\alpha = \frac{k_\alpha}{k}\mathbf{D}_\alpha\right) \tag{1.4}$$

$$x \to \infty: \ |\mathbf{U}^L| < \infty \tag{1.5}$$

where $\mathbf{U}_0^F(x, k_\alpha)$ is the Fourier transform of the function $\mathbf{U}_0(x, y_\alpha)$. Further analysis is conducted for fixed values of the wave-vector components $\{k_\alpha\}$.

The characteristic matrix of the system of equations (1.3) has the form

$$\mathbf{M}_x = \mathbf{A}^{-1}(k\mathbf{B} - k\Omega\mathbf{E}) + k\lambda\mathbf{E}$$

The roots λ_j of the characteristic equation $\det\mathbf{M}_x = 0$ are independent of k because system (1.1) is homogeneous with respect to the order of differentiation. Since system (1.1) is hyperbolic, it follows that as $\Omega \to \infty$ the matrix $\mathbf{M}_x$ has n linearly independent eigenvectors $\mathbf{l}^{(j)}$ that correspond to the roots $\lambda = \lambda_j$ which are real for real Ω. For $\Omega = O(1)$, the roots λ_j may become complex and there exists a set of isolated values Ω_0^0 such that $\lambda_i(\Omega_0^0) = \lambda_j(\Omega_0^0)$ and $\mathbf{l}^{(i)}(\Omega_0^0) \| \mathbf{l}^{(j)}(\Omega_0^0)$. Therefore, for all other Ω the solution of system (1.3) obtained by the method of variation of constants is written in the form

$$\begin{gathered} \mathbf{U}^L = \sum_{j=1}^{n} c_j^0 \mathbf{l}^{(j)} e^{i\lambda_j kx} + \sum_{j=1}^{n} \mathbf{l}^{(j)} e^{i\lambda_j kx} R_j(x) \\ R_j(x) = \int_{x_j}^{x} [\mathbf{L}^{-1}\mathbf{A}^{-1}\mathbf{U}_0^F]_j\, e^{-i\lambda_j k\xi}\, d\xi \end{gathered} \tag{1.6}$$

Here $\mathbf{L}$ is the $n \times n$ matrix whose columns are the eigenvalues $\mathbf{l}^{(j)}$ and c_j^0 are arbitrary constants to be determined from conditions (1.4), (1.5). The constants x_j will be chosen later.

All the roots of the characteristic equation are divided into two groups: the first group contains the roots $\{\lambda_q\}$: $\operatorname{Im}\lambda_q > 0$ ($q = 1, 2, \ldots, s$) and the second group the roots $\{\lambda i\}$: $\operatorname{Im}\lambda_i < 0$ ($i = s+1,$ $s, +2, \ldots, n$). This grouping is done for $\operatorname{Im}\Omega \gg 1$, i.e. in that part of the half-plane Ω which contains the integration path for the Laplace transform. Note that in order to separate the roots in the entire Ω-plane, we need to pass cuts $I_{q,i}$ between the branching points of the roots of the first and the second groups. These points form the set (Ω_0) and are part of the previously introduced set $\{\Omega_0^0\}$ of branching points of the multivalued function $\lambda(\Omega)$.

We will show that the set $\{\Omega_0\}$ lies on the real axis. Since system (1.1) is hyperbolic and homogeneous with respect to the order of differentiation, its solutions have the stability property: $\operatorname{Im}\lambda \neq 0$ for $\operatorname{Im}\Omega > 0$. The branching points of the roots from different groups therefore do not lie in the upper Ω half-plane. The problem is reversible in time, which corresponds to a simultaneous sign change of Ω, k_α and λ; therefore the branching points do not lie in the lower Ω half-plane either.

Substituting (1.6) into (1.5), we find that $c_i^0 = 0$ ($i = s+1, \ldots, n$), and the constants c_q^0 ($q = 1, \ldots, s$) are determined from system (1.5). A necessary condition for the problem to be well-posed is thus the equality $s = m$ [11]. The final result can be written in the form

$$\mathbf{U}^L(x, k_\alpha, \Omega) = \sum_{q=1}^{m} c_q^\circ \mathbf{l}^{(q)} e^{i\lambda_q kx} + \sum_{j=1}^{n} \mathbf{l}^{(j)} e^{i\lambda_j kx} R_j \tag{1.7}$$

$$\mathbf{c}^\circ = \mathbf{W}_1^{-1}[\mathbf{W}_2\mathbf{R}_0 + i\mathbf{V}^\circ\mathbf{U}_0^F(0, k_\alpha)], \quad \mathbf{c}^\circ = \{c_q^\circ\}, \ q = 1, 2, \ldots, m$$

$$(\mathbf{W}_1, \mathbf{W}_2) \equiv \mathbf{W}, \quad W_{qg} = \sum_{j=1}^{n} G_{qj}^{(g)} l_j^{(g)}, \quad g = 1, 2, \ldots, n; \ q = 1, \ldots m$$

$$\mathbf{G}^{(g)} = -\Omega\mathbf{E}^1 + \lambda_g\mathbf{C} + \mathbf{D}, \quad \mathbf{V}^0 = -\mathbf{E}^1 + \mathbf{V}\mathbf{L}^{-1}\mathbf{A}^{-1}$$

$$V_{qp} = \sum_{j=1}^{n} C_{qp} l_j^{(p)}, \quad q = 1, 2 \ldots, m, \quad p = 1, 2, \ldots, n$$

$$R_j(x) = \int_{x_j}^{x} [\mathbf{L}^{-1}\mathbf{A}^{-1}\mathbf{U}_0^F(\xi, k_\alpha)]_j e^{i\lambda_j k\xi} d\xi$$

$$R_{0j} = R_j|_{x=0}, \quad x_j = \begin{cases} 0, & j = 1, \ldots, m \\ \infty, & j = m+1, \ldots, n \end{cases}$$

The matrix $\mathbf{W}_1$ is formed by the first m columns of the matrix $\mathbf{W}$ and corresponds to waves moving away from the boundary. The matrix $\mathbf{W}_2$ formed by the next $n-m$ columns of $\mathbf{W}$ corresponds to arriving waves.

Now, inverting the Laplace transform and shifting the integration contour down in the Ω-plane as shown in Fig. 1, we obtain an asymptotic expression for the solution for $t \to \infty$, $x = \text{const}$:

$$\mathbf{U}^F(t, x, k_\alpha) = \frac{1}{2\pi i} \sum \operatorname{res} \mathbf{U}^L e^{-ik\Omega t} + \frac{1}{2\pi i} \sum \int_{I_h} \mathbf{U}^L e^{ik\Omega t} d\Omega \tag{1.8}$$

The summation is over all residues of the function $\mathbf{U}^L(\Omega)$ and all cuts I_h passed between the branching points of the roots from different groups (a cut is shown in Fig. 1 by a dashed line). On the right-hand side, we have omitted the rapidly decaying integral over the horizontal part of the integration contour. We retain in (1.8) only the integrals over the cuts between the branching points of the roots from different groups, because these roots occur in the solution of the problem in a different form (the matrix $\mathbf{W}_1$ contains only the first-group roots, while the matrix $\mathbf{W}_2$ contains only second-group roots).

Therefore, the solution $\mathbf{U}^L$ changes as we move around the branching points of the roots from different groups. When we circle the branching point of the roots from the same group, only the indexing of the roots within the group changes and $\mathbf{U}^L$ remains unchanged.

2. CONTRIBUTION OF THE INTEGRALS OVER THE CUTS TO THE ASYMPTOTIC BEHAVIOUR

Let us study the asymptotic behaviour as $t \to \infty$ of the integrals over the cuts I_h in equality (1.8). The asymptotic behaviour as $t \to \infty$ of the integrals over the cuts is determined only by the top-most parts of the integration contours. Therefore, each branching point makes an independent contribution to the asymptotic behaviour. If the two values of the function $\lambda(\Omega)$ are equal at a branching point, then the integrand $\mathbf{U}^L(\Omega)$ (the arguments k_α and x are assumed constant and therefore omitted) can be represented in the form

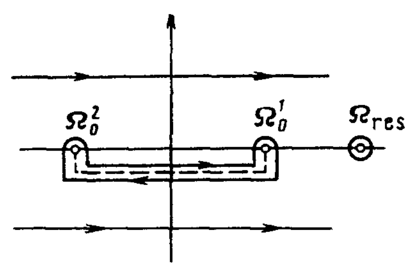

FIG. 1.

$$\mathbf{U}^L(\Omega) = \mathbf{U}_0^L(\Omega) + \mathbf{U}_1^L(\Omega)\sqrt{\Omega - \Omega_0}$$

where Ω_0 is a branching point and $\mathbf{U}_0^L$ and $\mathbf{U}_1^L$ are analytical functions. In the general position, the function $\mathbf{U}_1^L$ takes a finite value for $\Omega = \Omega_0$ (this can be obtained from (1.7); for gas dynamics, see [2]), and integration over the cut in the neighbourhood of the branching point produces the principal asymptotic term of the form

$$\mathbf{U}^F(t) \sim ct^{-3/2}e^{-ik\Omega_0 t}, \qquad c = 2\mathbf{U}^L(\Omega_0)\int_0^{-i\infty} e^{-i\xi}\sqrt{\xi}\,d\xi, \qquad \xi = (\Omega - \Omega_0)t$$

If the function $\mathbf{U}_1^L(\Omega)$ behaves as $(\Omega - \Omega_0)^{-1}$, which is possible in singular cases, when the pole and the branching point of $\mathbf{U}^L(\Omega)$ coincide, then $|\mathbf{U}^F(t)| \sim ct^{-1/2}\exp(-ik\Omega_0 t)$. These versions of the asymptotic behaviour match the results of [2] for gas dynamics.

We note without derivation that along the rays $x/t = \text{const}$ the asymptotic behaviour of the solution is determined by the saddle point (as in the case without a boundary) and the time dependence of the principal asymptotic term in the general position is typical of dispersing waves: $|\mathbf{U}^F(t)| \sim t^{-1/2}$ (for $k_\alpha = \text{const}$ the waves have a dispersion along x). We thus conclude that the existence of a boundary produces additional damping of the waves that propagate along the surface of discontinuity and this in general leads to the asymptotic behaviour $|\mathbf{U}^F(t)| \sim t^{-3/2}$.

The explanation of this effect is that for disturbances propagating along the boundaries (and it is these disturbances that correspond to the points Ω_0, see below) the reflection coefficient is -1. This is attributable to the equality at the branching point of the eigenvectors of the matrix $\mathbf{M}_x$ that correspond to the incident and the reflected waves. The disturbance reflected from the boundary is jointly annihilated with the incident disturbance in the principal term.

3. THE CONTRIBUTION OF RESIDUES TO THE ASYMPTOTIC BEHAVIOUR

The residues of the function $\mathbf{U}^L(\Omega)$, as follows from (1.7), are the zeros of the determinant $D_w = \det \mathbf{W}_1$. Having passed the cuts I_h, we can select the single-valued branch of the function $D_w(\Omega)$, which is obtained by continuation from the upper Ω half-plane. It is this single-valued branch that is considered in what follows.

Let us investigate its behaviour on the real axis Ω. It follows from (1.7) that the function D_w is real-valued on the sections of the real axis where all λ corresponding to outgoing waves are real. On the other hand, on the sections of the real axis Ω where at least one pair of complex-conjugate roots λ exists (these roots always belong to different groups, see Sec. 1), the function D_w is complex-valued.

This follows from the fact that, for any pair of complex conjugate roots, only one root corresponds to outgoing waves, and this root is included in D_w. In the general position, $\operatorname{Im} D_w \neq 0$ since all the coefficients in D_w are real.

As we have noted before, the hyperbolic type of the system implies that for large real Ω all λ_j and $D_w(\Omega)$ are real. By the Schwarz theorem of analytical continuation, the function $D_w(\Omega)$ takes complex-conjugate values at complex-conjugate points. If this function has complex roots, the solution $\mathbf{U}^F(\rightarrow)$ contains a component that exponentially increases with time with a growth rate that tends to infinity as $k \rightarrow \infty$, i.e. problem (1.1), (1.2) is ill-posed. If $D_w(\Omega)$ has only real roots, then the vector $\mathbf{U}^F(t)$ is bounded as $t \rightarrow \infty$ and in general does not tend to zero (neutral stability).

4. THE CONDITIONS FOR A TRANSITION BETWEEN DIFFERENT TYPES OF ASYMPTOTIC BEHAVIOUR OF THE SOLUTIONS

We will now give a geometrical interpretation of the results. To this end, consider the group velocity diagram, which describes the propagation of a weak shock from a point source after a unit

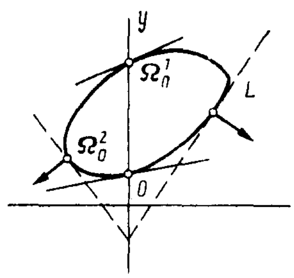

FIG. 2.

of time. Assume that the source is at the origin O (Fig. 2), the y-axis corresponds to the position of the boundary $x = 0$, and the curve L is the group velocity diagram. The figure shows the part of the diagram in the region $x<0$ (arriving waves).

Consider some point $y = \Omega$ on the y-axis sufficiently far from the point O. Each tangent drawn from Ω to L gives the direction of propagation of the wave corresponding to the chosen Ω [Ω is the quantity from Eq. (1.3)]. The root $\lambda(\Omega)$ of the characteristic equation corresponding to this wave is real. With a complex $\lambda(\Omega)$, it is impossible to construct the corresponding tangent from the point Ω to L. If the tangency point ($\Omega_0{}^2$ in Fig. 2) lies to the right (to the left) of the y-axis, the corresponding wave is an outgoing (arriving) wave with respect to the boundary.

As we have noted above, all λ are real for large Ω, i.e. n tangents to the group velocity diagram may be drawn from the point $y = \Omega$. The branching points of the function $\mathbf{U}^L(\Omega)$ arise only in cases when Ω coincides with $\Omega_0{}^1$—the point of intersection of the group velocity diagram with the y-axis.

Indeed, when the roots λ coincide, the directions of the tangents drawn from the corresponding point $\Omega = \Omega_0{}^1$ also coincide. If the tangency points do not coincide, then in the general position the eigenvectors characterizing the waves that correspond to these points are different, and the solution $\mathbf{U}^L(\Omega)$ does not have singularities in the neighbourhood of the point $\Omega_0{}^1$. On the other hand, the tangency points may coincide only on the y-axis, because group velocity diagrams do not have points of inflection. The last assertion is a consequence of the fact that for a hyperbolic system each direction of the normal corresponds to the same number of characteristic velocities (which is equal to the order of the system).

Thus, on the real axis Ω, the number of real roots $\lambda(\Omega)$ may change only at the point $\Omega = \Omega_0{}^1$ that correspond to the intersection of the group velocity diagram with the y-axis. These points delimit sections of the real axis Ω where all $\lambda(\Omega)$ are real and they are the branching points of the function $\mathbf{U}^F(\Omega)$.

Let us consider how the complex roots of $D_w(\Omega)$ can be shifted to the real axis by changing the system parameters. When the last complex root reaches the real axis, the system passes from an unstable to a stable or a neutrally stable state.

A root of $D_w(\Omega)$ may reach the real axis either at infinity or at a finite point of the axis. We will start with the first case. Let $\Omega \to \infty$ as $H \to H^0$, where H is a vector in the space of the system parameters. Then, by the hyperbolicity of system (1.1), $\lambda_j \sim a_j\Omega$, where a_j are the reciprocals of the characteristic velocities, and D_w is reduced to a polynomial:

$$D_w(\Omega) \sim b_0\Omega^m + b_1\Omega^{m-k} + \dots = 0$$

where $b_0 \to 0$ as $H \to H^0$. Then as $H \to H^0$ we obtain $\Omega^k = -b_1/b_0$, $b_0 \to 0$.

In the general position, $\partial b_0/\partial H_i^0 \neq 0$ for $H = H^0$ and therefore b_0 changes its sign in the neighbourhood of the point H^0 as we cross the surface Σ_0 defined by the equation $b_0(H_i) = 0$. Therefore, Ω remains real for $k = 1$ on both sides of the surface Σ_0 and the complex-valued root does not go to the real axis. If $k = 2$, then $\Omega = \pm i(b_1/b_0)^{1/2}$ for $b_1/b_0 > 0$ and Ω is real for $b_1/b_0 < 0$. Thus, for $k = 2$, we have a transition from instability to neutral stability. For $k = 2$, the surface Σ_0 is the boundary between the zones of instability and neutral stability in the space of system parameters. If $k \geqslant 3$, then regardless of the sign of b_1/b_0, for H close to H^0 the function $D_w(\Omega)$ has a zero in the upper half-plane, i.e. the type of solution is preserved.

Note that the case $k = 2$ is exceptional, because in general $k = 1$. However, there is an important group of applied problems which are invariant under the change of y to $-y$ (gas-dynamic shocks and some special cases of MHD shocks). In these problems, $D_w(\Omega)$ is an even function.

Now suppose that as $H \to H^0$ the complex-conjugate roots of the function D_w reach the real axis Ω at a finite point Ω_0 that lies outside the cuts I_h, which are now passed on the real axis Ω between branching points. Then $D_w(\Omega)$ has a multiple root at this point, and this root is of multiplicity two in the general position. As the parameters are changed further, the roots become real. The surface Σ_1 in the space H on which this happens is another boundary between instability and neutral stability.

If the root crosses from the complex plane to the real axis on some cut I_h, then further change of the parameters causes the root to escape from the given sheet of the Riemannian surface and it ceases to make a contribution to the solution.

Yet another possibility of the appearance of a real zero of the function $D_w(\Omega)$ is when the root moves from the cut to the real axis. At the instant it reaches the real axis, the zero (a simple zero in the general position) coincides with a branching point of the roots of the characteristic equation. The coincidence of the zero with a branching point may constitute a boundary (in the parameter space, this is some surface Σ_2) between stability [when $D_w(\Omega)$ does not have roots on the given sheet of the Riemannian surface] and neutral stability [when the roots of $D_w(\Omega)$ are on the real axis]. It is in this case that we obtain the asymptotic behaviour $|\mathbf{U}^F| \sim t^{-1/2}$ noted in Sec. 2.

It follows from these results that the region of neutral stability O_N is of the same dimensions as the parameter space of problem (1.1), (1.2). Therefore, for each interior point of parameter space, problem (1.1), (1.2) is stable, i.e. small changes in the coefficients of system (1.1) and the boundary conditions (1.2) for $x = 0$, $t \geqslant 0$ do not produce a solution of a new type.

5. THE WELL-POSED FORM OF PROBLEMS, NON-HOMOGENEOUS WITH RESPECT TO THE ORDER OF DIFFERENTIATION

If we modify the formulation of our problem by introducing additional non-differential terms into Eqs (1.1) and the boundary conditions (1.2), the problem will remain well-posed if the original problem was well-posed. Indeed, since the ill-posed properties may manifest themselves for large $k\Omega$, when the additional terms are small, the increment $\Delta\Omega$ of the root Ω_i of the equation $D_w(\Omega) = 0$ is obtained from the equality

$$a\,(\Delta\Omega)^f + bk^{-1} = 0$$

where the first term is the principal part of the increment $D_w(\Omega_i + \Delta\Omega) - D_w(\Omega)$ (f is the multiplicity of the root Ω_i) and the second term is the value for $\Omega = \Omega_i$ of the additional terms that occur in the equation for Ω when the problem is modified as suggested above. The factor k^{-1} is associated with the lower order of differentiation of the additional terms compared with the original terms.

The transition to an ill-posed problem (and hence to instability) due to the appearance of new terms is possible only when Ω_i is a real root. If it is simple ($f = 1$), then $\Delta\Omega$ is of the order k^{-1}, and the corresponding increment $k\Delta\Omega$ is bounded as $k \to \infty$. Thus, the problem remains well-posed in this case, but instability with a bounded growth rate may arise. This conclusion does not apply for $f \geqslant 2$; however, as we have seen, the presence of a multiple real root Ω corresponds to a boundary of the stability region in parameter space.

Instability or ill-posed behaviour may also develop as a result of the displacement of a branching point of the roots $\lambda(\Omega)$ from the real axis to the upper half-plane due to the appearance of non-differential terms. In practice, this does not occur, however.

Let us first consider a simple branching point Ω_0, where the two branches of the function $\lambda(\Omega)$ have equal values. Considering a small neighbourhood of the point Ω_0 and taking small changes $\Delta\lambda(\Omega)$, we need to examine only a quadratic equation for $\Delta\lambda$ with coefficients that depend on Ω. At the point $\Omega = \Omega_0$, the discriminant should vanish. In the simplest case, in a small neighbourhood of the point Ω_0 the discriminant may be viewed as a linear function of Ω. Then both roots $\lambda(\Omega)$ are real on one side of the branching point and both are complex on the other side. In this case, the allowance for additional terms containing k^{-1} will shift the branching point Ω by $\Delta\Omega \sim k^{-1}$, which may only result in a bounded growth rate of the solution, i.e. instability but not ill-posed behaviour.

The conclusion is false if the discriminant of the original problem does not contain a linear term and starts with a quadratic term. In this case, however, the point Ω_0 may be regarded as the result of confluence of two branching points. It follows from Sec. 4 that in this case the group velocity diagram is tangent to the boundary, i.e. the velocity of one of the weak shocks is zero in the coordinate system attached to the boundary. This corresponds to the boundary of the region of an evolving (and hence well-posed) shock. The cases when the function $\lambda(\Omega)$ has real branching points of multiplicity greater than two have not been considered.

6. LOSS OF SMOOTHNESS DUE TO DISTURBANCE IN REFLECTION FROM A NEUTRALLY STABLE BOUNDARY

One of the important properties of the solution of problem (1.1), (1.2) in the case of neutral stability is the loss of smoothness associated with the appearance of one or several real zeros of the function $D_w(\Omega)$ that generate real poles of the solution $\mathbf{U}^L(\Omega)$. For these cases, we do not have bounds that ensure the same smoothness for the solution as for the initial data [8, 9]. Below we propose a simple, though non-rigorous, explanation of the phenomenon of loss of smoothness.

The loss of smoothness is easily noted in the asymptotic form of the solution (1.8). The poles of the function $\mathbf{U}^L(\Omega)$ obviously make a contribution proportional to k in the evaluation of the inverse Laplace transform with respect to time (integration over $\omega = k\Omega$):

$$\int \frac{A(k,\omega)}{(\omega/k-\Omega_0)} e^{i\omega t}\,d\omega = kA(k, k\Omega_0)\,e^{ik\Omega_0 t}$$

As a result, the function that corresponds to the residue and represents the response of the boundary will tend more slowly to zero a $k \to \infty$ than the function $A(k,\omega)$ that represents incident disturbances.

We will now propose a different interpretation of this issue, which makes it possible to trace the process over time. Since each pole isolates a certain value of Ω_0, and the part of the solution associated with the residue is obtained as an integral over an arbitrarily small circle around the point Ω_0, the latter implies that this part of the solution is actually a sum of n plane waves corresponding to real $\lambda(\omega_0)$.

Assume that there is at least one arriving wave (u) and one outgoing wave (v). This restriction is unimportant and is used only for simplicity. The boundary condition on $x = 0$ is written in the form

$$\frac{\partial v}{\partial t} + \Omega_0 \frac{\partial v}{\partial y} = A\frac{\partial u}{\partial t} + B\frac{\partial u}{\partial y} \tag{6.1}$$

Here we assume that the derivatives with respect to x have been eliminated from the boundary conditions using equations for u and v. The function $D_w(\Omega_0)$ defined by the operator on the left-hand side of the equality (6.1) has a zero at $\Omega = \Omega_0$. The function $u(t, y)$ is assumed to be known. Passing to a coordinate system that moves along the y-axis with velocity Ω_0, we obtain from (6.1) in this system that the partial derivative of v with respect to t is expressible linearly in terms of $\partial u/\partial t$ and $\partial u/\partial y$. Hence we see that the y-smoothness of the outgoing disturbance v may be an order of magnitude less than the smoothness of the arriving disturbance u.

Note that a similar result follows directly from relationships (1.7), (1.8) after taking inverse Fourier–Laplace transformations.

7. ON THE POSSIBILITY OF DESCRIBING DISTURBANCES ON THE BOUNDARY BY A DIFFERENTIAL EQUATION

Note that the left-hand side of the equation $D_w(\omega, k) = 0$, where $\omega = k\Omega$, defining the eigenvalues of problem (1.1), (1.2) is not a polynomial in ω and k, because it contains the quantities $\lambda(\omega, k)$ that depend in a complicated manner on their arguments and D_w includes only λ that correspond to outgoing waves. Therefore, a partial differential equation cannot be associated with the above equation. However, some pseudodifferential equation satisfied by the solution **U** on the boundary can be associated with this equation:

$$D_w^1\,(\partial/\partial t,\ \partial/\partial y)u = f\,(y,\ t) \tag{7.1}$$

where u is any of the unknown functions, f describes the effect of the disturbances arriving on the boundary and the function D_w^1 is not a polynomial in its arguments, i.e. it is not a differential operator: it is only understood in the sense that the Fourier–Laplace transformation of D_w^1 produces $D_w(\omega, k)$. If all the roots λ_j occur in some expression symmetrically, then by a well-known theorem in algebra [12] such symmetric algebraic functions of λ_j can be expressed in terms of the coefficients of the equation satisfied by λ, i.e. in terms of polynomials of ω and k. Therefore, if we act on both sides of (7.1) by the product of the operators D_{wi}^1, which are identical with D_w^1 except that each contains other roots λ_j so that the product is symmetrical in all roots, we obtain

$$D\,(\partial/\partial t,\ \partial/\partial y) \equiv D_{w1}^1 D_{w2}^1 \ldots D_{wN}^1 D_w^1 u = D_{w1}^1 D_{w2}^1 \ldots D_{wN}^1 f \tag{7.2}$$

where, by the above argument, the operator $D(\partial/\partial t,\ \partial/\partial y)$ is a polynomial in its arguments, i.e. a differential operator. Thus, the original mixed boundary-value problem (1.1), (1.2) has been reduced to a Cauchy problem for one differential equation (7.2) of a high order. For gas dynamics, this result has been obtained by a different method in [10].

Note the special structure of the right-hand side of equality (7.2), which ensures that the solutions of Eqs (7.1) and (7.2) correspond. If we consider a problem with non-zero initial conditions, then certain constraints should be imposed on these conditions for Eq. (7.2) so that the solution also satisfies Eq. (7.1).

Note that the use of Eq. (7.2) in practice is difficult for two reasons. First, the order of the resulting equation is very high: the number of factors in D is C_n^m, where m is the number of waves moving away from the boundary and n is the order of the system. Second, the function $D(\omega, k)$ has "redundant" zeros on other sheets of the Riemannian surface over the ω plane which are not needed for stability analysis. These zeros, however, do not have an effect due to the special form of the right-hand side of Eq. (7.2).

8. ON NEUTRALLY STABLE SHOCK WAVES IN MAGNETOHYDRODYNAMICS

As an application, consider the existence of neutrally stable magnetohydrodynamic shocks. We will consider only fast shock waves with the magnetic field perpendicular to the shock surface on both sides. The shock wave is fast if [13] $A^2 \equiv B^2/(4\pi\rho u^2) < 1$, where B is the magnetic field strength and u and ρ are the velocity and the density of the gas behind the shock.

It has been previously shown [5] that the magnetic field does not affect the stability criterion of such a shock wave: the equation for the disturbance eigenfrequencies is independent of B and has the form

$$[2M^2 - \delta - 1 - (\delta - 1)\sigma M^2]z^2 - 2\,(\delta - M^2)z + (\delta - 1)(\sigma - 1) = 0$$
$$z = \omega/\lambda - 1 \tag{8.1}$$

Here z is the perturbation frequency (in units of λ) in the coordinate system attached to the gas behind the shock, $\sigma \equiv \rho/\rho_0 > 1$ is the density ratio on the shock, $\delta = -(\rho u)^2(\partial 1/\rho/\partial p)_H$ is the dimensionless derivative along the shock adiabatic and $M < 1$ is the Mach number behind the shock. The gas velocity u is taken as the characteristic velocity.

For an ideal gas

$$\delta = \frac{1}{M_\infty^2}, \qquad \sigma = \frac{(\gamma+1)M_\infty^2}{2+(\gamma-1)M_\infty^2}, \qquad M^2 = \frac{2+(\gamma-1)M_\infty^2}{2\gamma M_\infty^2 - \gamma + 1} \tag{8.2}$$

where M_∞ is the Mach number of the incident flow. Note that z satisfies the dispersion equation for magnetosonic waves:

$$z^4 - \left(A^2 + \frac{1}{M^2}\right)\left(1 + \frac{k_y^2}{\lambda^2}\right)z^2 + \frac{A^2}{M^2}\left(1 + \frac{k_y^2}{\lambda^2}\right) = 0 \tag{8.3}$$

As we have shown above, the transition from stability to neutral stability occurs when the eigenfrequency equation (8.1) (the function D_w) has a root corresponding to a disturbance that separates the arriving and the outgoing waves, i.e. to a disturbance with zero x-component of the group velocity in the coordinate system attached to the shock wave:

$$\partial\omega/\partial\lambda = 0 \tag{8.4}$$

Eliminating k_y/λ from the relationships (8.2), (8.3), and $z = \omega/\lambda - 1$, we obtain

$$\left(A^2 + \frac{1}{M^2}\right)z^5 + \left(\frac{1}{M^4} + \frac{A^2}{M^2} + A^4\right)z^4 - \frac{2A^2}{M^2}z^3 - \\ - \frac{2A^2}{M^2}\left(A^2 + \frac{1}{M^2}\right)z^2 + \frac{A^4}{M^4} = 0 \tag{8.5}$$

For $A^2 = 0$, Eq. (8.5) has the root $z = -1/M^2$, which corresponds to gas-dynamic disturbances, and the four-fold root $z = 0$. Since the shock is fast, it may interact only with fast magnetosonic disturbances, which are continuously generated from gas-dynamic disturbances as A^2 is increased. For small A^2, we obtain for the relevant root from (8.5)

$$z = -M^{-2}(1 + \alpha), \quad \alpha = M^4(1 - M^2)A^4 \ll 1 \tag{8.6}$$

The condition for a transition from stability to neutral stability is that this solution equals the root of Eq. (8.1). Substituting (8.6) into (8.1), we obtain the value $\delta = \delta_1$ that corresponds to the boundary between stable and neutrally stable shock waves

$$\delta_1 = \delta_0 + c\alpha \tag{8.7}$$

$$\delta_0 = \frac{\sigma M^2 + M^2 - 1}{\sigma M^2 + 1 - M^2}, \qquad c = \frac{2M^2}{\sigma M^2 + 1 - M^2}$$

From (8.7) it follows that the magnetic field increases the region of neutral stability compared with the case when $B = 0$, when this region is located to the left of the point $\delta = \delta_0$.

Substituting (8.2) into (8.7), we find that in an ideal gas a strong ($M_\infty \gg 1$) shock transfers to neutral stability for a magnetic field strength

$$B \geqslant a_\infty \sqrt{\rho_\infty M_\infty} f(\gamma), \qquad f(\gamma) = \frac{8\pi\gamma^2}{(\gamma - 1)\sqrt{3\gamma^2 + 2\gamma - 1}}$$

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CALCULATION OF ROTATIONAL DERIVATIVES FOR "LOCAL" INTERACTION OF A FLOW WITH THE SURFACE OF A BODY†

A. I. BUNIMOVICH and A. V. DUBINSKII

Moscow

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The rotational derivatives of the force and moment characteristics are calculated for solids of revolution that move at an angle of attack with small angular velocity. Formulas for rotational derivatives of the second order are derived and analysed for the general class of "local" interaction models of the flow with the surface of the body.

THE DEVELOPMENT of analytical methods of calculation for rotational derivatives in the non-translational motion of bodies in free-molecular flow is considered in [1–3]; corresponding methods for the intermediate rarefied gas flow region are developed in [2, 4, 5]. The approach proposed in [6] is intended for a fairly general class of "local" models describing the interaction of the flow with a rotating body; the implementation of this approach has led to working formulas for first rotational derivatives [6, 7]. In this paper, the proposed approach is further developed for second rotational derivatives.

In the attached coordinate system x_1, x_2, x_3 shown in Fig. 1, the expression for the radius vector of a point on the surface of the body can be represented in the form

$$\mathbf{r} = \Phi(\rho)\mathbf{x}_1^0 + \rho\cos\theta\mathbf{x}_2^0 + \rho\sin\theta\mathbf{x}_3^0$$

where x_1^0, x_2^0, x_3^0 are the unit vectors of the coordinate axes; the function $\Phi(\rho)$ defines the generator of the solid of revolution with a plane maximum middle section of radius R, and

$$\Phi(0) = 0,\ \Phi'(0) > 0,\ \Phi''(\rho) > 0,\ 0 \leqslant \rho \leqslant R,\ \Phi'(R) < \infty$$

The axes are oriented so that the translational velocity vector $\mathbf{v}_\infty$ is in the x_1, x_2 plane making an angle $\pi - \alpha$ with the x_1 axis

$$\mathbf{v}_\infty = -\mathbf{v}_\infty \cos\alpha x_1^0 + \mathbf{v}_\infty \sin\alpha x_2^0$$

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